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GRAPH INTERPRETATION OF IRREDUCIBLE, REDUCIBLE, PERIODIC, AND APERIODIC PROPERTIES IN MARKOV CHAINS

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ABSTRACT

Markov chains are widely used stochastic models for representing dynamic systems in which future states depend only on short-term probabilistic transitions. Fundamental structural properties irreducibility, reducibility, periodicity, and aperiodicity are essential for analyzing long-term behavior, particularly the existence and stability of stationary distributions. These properties are traditionally examined through transition probability matrices; however, matrix-based analysis can be computationally intensive and less intuitive for large-scale systems. This study introduces a graph-theoretic representation in which states are modeled as vertices and positive transition probabilities as directed edges. Within this framework, irreducibility corresponds to strong connectivity, reducibility to multiple communication classes, and periodicity to the greatest common divisor of return times determined by cycle structures. The results establish a formal relationship between algebraic and graph-based characterizations, providing a clearer, more interpretable, and computationally efficient approach for studying the structural properties of Markov chains.

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1. INTRODUCTION

Stochastic processes are a widely used mathematical framework for modeling dynamic systems that evolve randomly over time. One of the most fundamental models of stochastic processes is the Markov chain, a discrete process in which the probability of transition to the next state is determined solely by the current state, unaffected by previous history [1]. This characteristic, known as the Markov property, underlies the use of Markov chains in various fields, such as queuing systems, communication networks, reliability analysis, production systems, economics, and modeling social and biological phenomena [2]

The theoretical study of Markov chains is closely related to the classification of states that form the state space [3]. Several important structural properties to analyze include irreducibility and reducibility, which relate to the connectivity between states, and periodicity and aperiodicity, which relate to the time pattern of a process's return to a particular state. Understanding these properties is crucial because they determine the long-term behavior of Markov chains, including the existence and uniqueness of stationary distributions and the rate of convergence to them [4], [5]. Conventionally, the identification of Markov chain properties is performed through an analytical approach based on transition probability matrices. This approach includes analyzing interstate accessibility, examining communication strength, and calculating state periods through multi-step transition probabilities [6] [7]. While this method is formal and mathematically rigorous, the computational complexity can increase significantly with the number of states, making it difficult to intuitively interpret the system's structure [8].

As an alternative approach, Markov chains can be represented as directed graphs, where each state is represented as a vertex and each positive transition probability is represented as a directed arc [9]. This graph representation allows for analysis of Markov chain properties from a graph theory perspective, specifically through the concepts of reachability, strongly connected components, directed paths, and cycles. Several previous studies have utilized graph representations to study the structure of Markov chains, particularly in identifying communication classes and simplifying the state space of large-scale systems [10]. However, studies that explicitly and systematically link irreducible and reducible properties, as well as periodic and aperiodic properties, with the corresponding graph characteristics are relatively limited. In fact, this relationship has the potential to provide a more intuitive, visual, and efficient analytical framework for determining the fundamental properties of Markov chains, particularly in the context of learning and analyzing complex systems [11].

Based on this description, this study aims to examine the relationship between the properties of Markov chains and the properties of graphs formed from transition probability matrices [12]. The research focuses on irreducible, reducible, periodic, and aperiodic properties, examining how each of these properties is reflected in the structure of directed graphs through the interconnectedness of nodes and the presence of cycles. It is hoped that the results of this study can provide a conceptual contribution to understanding Markov chains through a more intuitive and easily interpretable graph approach [13].

2. RESEARCH METHODE

Methods used in study This nature analytical-conceptual [14], with steps as following:

a Determine a number of example matrix opportunity transition Markov chain.

b Transforming every matrix opportunity transition to in form graph directional, with knot representing states and arcs directional represent opportunity transition worth positive [15].

c Analyze structure the graph formed based on draft theory graphs, such as:

  a) existence track and cycle,
  b) connectedness between knot,
  c) possibility returns to knot origin

d Linking characteristics graph the with characteristic classification corresponding Markov chain.

3. RESULT AND ANALYSIS

3.1 Irreducible

A Markov chain is said irreducible If all condition is at in One class communication, which means every condition can achieved from condition others. Example irreducible Markov chain must fulfil condition that each row of the matrix opportunity transition amount to one and all the elements worth between zero and one. According to with understanding Markov chain irreducible, given matrix opportunity transition from something Markov chain that has characteristic the as following.

$$P_1 = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} & 0 \\ \frac{1}{2} & \frac{1}{4} & \frac{1}{4} \\ 0 & \frac{1}{3} & \frac{2}{3} \end{bmatrix}$$

Matrix opportunity transition P_1 with set condition $\{0,\ 1,\ 2\}$ it is said nature irreducible if all over condition in set the can each other connected or is at in One class communication. A Markov chains are irreducible If every condition can accessible from condition others, as shown with opportunity transition $P_{ij}^{(n)} > 0$ For something n.

On the matrix opportunity transition P_1, state 1 can be accessed from state 0 because $P_{01} = \frac{1}{2} > 0$, and vice versa state 0 can be accessed from state 1 because $P_{10} = \frac{1}{2} > 0$. This shows that states 0 and 1 communicate with each other. Furthermore, state 2 can be accessed from state 1 because $P_{12} = \frac{1}{4} > 0$, and state 1 can be accessed again from state 2 because $P_{21} = \frac{1}{3} > 0$, so states 1 and 2 also communicate with each other.

Because the states 0 and 1 are mutually exclusive communicate, and conditions 1 and 2 are mutually exclusive communicate, then in a way transitive states 0 and 2 are also mutually exclusive communicate. With Thus, all condition $\{0,\ 1,\ 2\}$ is at in One class communication, so that Markov chain with matrix opportunity transition P_1 nature irreducible.

Next, the matrix opportunity transition P_1 can represented in form graph directional. Because there are three circumstances, then graph P_1 own three nodes each representing One state. The arc between knot formed based on opportunity displacement from condition i to condition j.

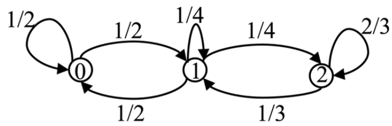

Figure 1: Transition Diagram P_1

Based on the transition diagram in Figure 1, it can be observed channel arc go out from every node that represents condition system. The node that represents state 0 has two bows out, that is A bracelet (self-loop) which shows possibility system still is in state 0, and A arc connecting direction state 0 to state 1.

The node that represents state 1 own three arc out, that is A bracelet that allows system still is in state 1, a arc directional from state 1 to condition 2, and A arc directional from state 1 to state 0. With existence bows said, state 1 has access direct to state 0 and state 2, as well can return to himself itself. The node that represents state 2 has two bows out, that is A bracelet that shows possibility system still is in state 2, and A arc connecting direction state 2 to state 1.

From the structure graph the can observed that state 0 can achieved from state 1 and state 2, as well from himself alone. Condition 1 can achieved from state 0 and state 2, as well from himself alone. Condition 2 can be achieved from state 1 and through track No directly, from state 0. With thus, for every partner condition i And j, there is a path directed from i to j and from j to i. This shows that all the vertices in the graph each other communicate (mutually accessible) and graphs the transition connected in a way strong.

3.2 Reducible

A Markov chain is said reducible if there is conditions that are not can connected One each other or No is at in One class communication. Markov chains that have more from One class communication show that no all condition each other communicate. The following served example Markov chain which is reducible with provision that amount elements in each row of the matrix opportunity transition worth one and the elements is at the zero interval until one. Based on understanding Markov chain which is reducible, given something matrix opportunity transition as following:

$$P_2 = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} & 0 \\ \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Matrix opportunity transition P_2 with set condition $\{0,\ 1,\ 2\}$ said to be reducible Because no all over condition can each other accessible or is at in One class communication. A condition it is said reducible if there is partner condition iAndj so that opportunity transition $P_{ij}^{(n)} > 0$ For every n. Condition 1 can accessible from state 0 because $P_{01} = \frac{1}{2} > 0$, and vice versa state 0 can accessible from condition 1 because $P_{10} = \frac{1}{2} > 0$. This shows that state 0 and state 1 communicate with each other. However, state 2 cannot be accessed from either state 0 or state 1 because $P_{20} = P_{21} = 0$, so condition 2 no communicate with state 0 and state 1.

Although state 0 and state 1 exist in One class communication, presence state of 2 separate cause third condition the No is at in One same class. Therefore that is, the Markov chain with matrix opportunity transition P_2 nature reducible. Furthermore, the matrix opportunity transition P_2 represented in form graph directional.

Because there are three states, graph P_2 own three nodes each representing One circumstances, with formed bow based on opportunity displacement from condition ito state j. Transition diagramP_2 shown in Figure 2.

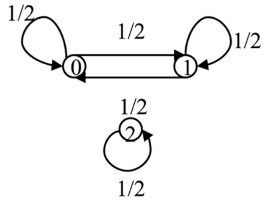

Figure 2: Transition Diagram P_2

Based on the transition diagram in Figure 2, it can be observed that state 0 and state 1 are mutually exclusive access, namely state 0 can achieved from state 1 and vice versa. With Thus, state 0 and state 1 exist in One class communication. On the other hand, condition 2 no can accessible from state 0 or state 1, and likewise no there is track directional from state 2 towards state 0 or condition 1. This is show that state 2 forms class communication alone separate from class communication that contains state 0 and 1.

Reviewed from channel arc out, the node that represents state 0 has two arcs out, that is A bracelet that shows possibility system still is in state 0, and A arc directional going to state 1. The vertex in state 1 also has two arcs out, that is A bracelet that allows system still are in states 1 and a arc directional going to state 0, so that state 0 and state 1 form something connected subgraph in a way strong. Meanwhile that is, the node is in state 2 only own One arc go out in the form of bracelet that connects state 2 to himself alone, without existence the arc that connects them with state 0 or state 1. Structure graphs the show that no all condition in mutually exclusive Markov chains communicate, so that graph transition reducible No connected in a way strong and composed on more from One class communication.

3.3 Periodic

A Markov chain is said periodic if there is a number of tracks from something condition iwhich returns to that state, with the number of steps on each path being a multiple of an integer $d(i) > 1$. In this condition, the state is said to have iperiodic properties. with period d. The following is an example of a periodic Markov chain, with the condition that the sum of the elements in each row of the transition probability matrix is 1 and each element is in the interval from zero to one. Based on this definition, a transition probability matrix representing a periodic Markov chain is given as follows.

$$P_3 = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ \frac{1}{2} & 0 & \frac{1}{2} & 0 \end{bmatrix}$$

Matrix opportunity transition P_3 with set condition $\{0,\ 1,\ 2\}$ it is said nature periodic if every condition in set the own period $d(i)$ which is factor fellowship the biggest from all over number n so that $P_{ii}^{n} > 0$. A condition called *periodic* if $d\,(i) > 0$, with period defined as $d\,(i) = FPB\,\{n \geq 1 | P_{ii}^{n} > 0\}$. Based on the results of multiplying the transition probability matrices, the probability that the state ireturns to the state iat each particular step is obtained. From these results, it can be concluded that state 0 has *periodic properties.* with period 2. Likewise, state 1, state 2, and state 3 are also characteristic periodic with period 2.

Next steps is represent matrix opportunity transition P_3 to in form graph. Matrix P_3 with set condition $\{0,\ 1,\ 2, 3\}$ represented as a graph P_3 which consist on four nodes, where each knot represent One circumstances. Arcs in a graph formed based on opportunity displacement from condition i to condition j. Transition diagram for transition probability matrix P_3 as following.

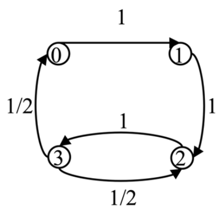

Figure 3: Transition Diagram P_3

Based on the transition diagram in Figure 3, it can be analyzed long possible trajectory every condition return to condition originally. For state 0, system can return to condition beginning in the 4th step through track $0 \to 1 \to 2 \to 3 \to 0$, as well as in the 6th step through the trajectory $0 \to 1 \to 2 \to 3 \to 2 \to 3 \to 0$. State 1 returns to its original state in the 4th step through track $1 \to 2 \to 3 \to 0 \to 1$, and in the 6th step through the path $1 \to 2 \to 3 \to 2 \to 3 \to 0 \to 1$. Circumstances 2 can return to state 2 in the 2nd step through track $2 \to 3 \to 2$, in the 4th step via the path $2 \to 3 \to 0 \to 1 \to 2$, and in the 6th step via the path $2 \to 3 \to 0 \to 1 \to 2 \to 3 \to 2$. Next, state 3 return to condition return to step 2 through track $3 \to 2 \to 3$, in the 4th step through the path $3 \to 0 \to 1 \to 2 \to 3$ and in the 6th step through the path $3 \to 2 \to 3 \to 0 \to 1 \to 2 \to 3$. With thus, a knot in Markov chain is said periodic if factor fellowship the biggest from all over long track direction that allows return to knot origin worth bigger from one. In graphics, a knot periodic if structure graph the transition only form cycles with long steps which are multiple from something number round certain.

3.4 Aperiodic

A Markov chain is said aperiodic if there is track from something condition iwho returned to condition the with amount step from every channel is multiple number round, so that period condition the worth $d(i) = 1$. Following example Markov chain which is aperiodic, with provision that amount elements in each row of the matrix opportunity transition worth one and the elements is at the zero interval until one. Based on definition said, given something matrix opportunity transition that represents aperiodic Markov chain as following.

$$P_4 = \begin{bmatrix} \frac{1}{2} & \frac{1}{4} & \frac{1}{4} \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}$$

Matrix opportunity transition P_4 with set condition $\{0,\ 1,\ 2\}$ it is said nature aperiodic if every condition in set the own period $d(i)$ which is factor fellowship the biggest from all over number n so that $\mathrm{P}_{\mathrm{ii}}^n > 0$. A condition is called aperiodic if $d(i) = 1$, with the period defined as $d(i) = FPB\{n \geq 1 | \mathrm{P}_{\mathrm{ii}}^n > 0\}$.

Based on results multiplication matrix opportunity transition, obtained opportunity return every condition i to condition originally on several step certain. For state 0, obtained $FPB\{1,2,3,4,5\} = 1$, so that state 0 is aperiodic. Furthermore, for state 1 is obtained $FPB\{3,4,6\} = 1$, which shows that state 1 is also aperiodic. State 2 is obtained $FPB\{2,3,4,5\} = 1$, so that state 2 is aperiodic.

Stage furthermore is represented matrix opportunity transition P_4 into a graph. A matrix P_4 with a set of states 0,1,2 is depicted as a graph P_4 consisting of three vertices, where each vertex represents one state. The arcs in the graph are formed based on the probability of moving from state ito state j. The transition diagram for the transition probability matrix P_4 is shown as follows.

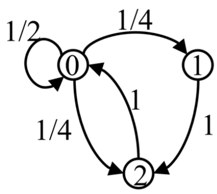

Figure 4: Transition Diagram P_4

Aperiodic properties something nodes in a Markov chain are determined by values $d(i)$, which is the greatest common factor of all the numbers nso that $\mathrm{P}_{\mathrm{ii}}^n > 0$. If obtained $d(i) = 1$, then the node is said to be aperiodic. Markn=1 including in set the If there is opportunity return to condition back to the beginning in One step, namely $\mathrm{P}_{\mathrm{ii}}^1 > 0$.

Based on the transition diagram in Figure 4, the node that represents state 0 own arc in the form of bracelet (self-loop), so that state 0 can return to himself Alone in One step through track $0 \to 0$. With thus,n=1 including in set long track return to state 0. Next, state 1 can return to condition back to the beginning through a number of track with long different steps, including in step 3 through track $1 \to 2 \to 0 \to 1$, in the 4th step through the path $1 \to 2 \to 0 \to 0 \to 1$ and in the 5th step through the path $1 \to 2 \to 0 \to 2 \to 0 \to 1$. Meanwhile That, state 2 can return to condition return to step 2 through track $2 \to 0 \to 2$, in the 3rd step through the path $2 \to 0 \to 1 \to 2$ and $2 \to 0 \to 0 \to 2$, and in the 4th step through the path $1 \to 2 \to 0 \to 0 \to 1$.

From the tracks the seen that every knot own possibility returns to condition back to the beginning in various long track, good odd and even. Therefore that, factor fellowship the biggest from all over mark n that fulfills $\mathrm{P}_{\mathrm{ii}}^n > 0$ is $d(i) = 1$. Thus, all nodes in the Markov chain represented by Figure 4 are aperiodic. In general, a knot in Markov chain is said aperiodic if graph the transition allows return knot the through tracks with long steps that are not own factor fellowship the biggest more from one. If at a time knot there is bracelet (*self-loop*), then knot the Certain nature aperiodic, because allows return knot to condition back to the beginning in One step. In addition, a knot can also nature aperiodic although no own bracelet, during graph the transition allows existence track return to knot the with long mutual steps relatively prime.

4. CONCLUSION

Based on analysis structure graph transition, nature irreducible and reducible in a Markov chain is determined by the connectedness between states. Markov chains are said to irreducible if all over condition each other communicate, namely for every partner condition there is track two- way direction, so that graph the transition connected in a way strong. Instead, a Markov chain is said reducible if no all condition each other access, which is marked with existence more from One class communication on graphs transition.

Temporary that, nature periodic and aperiodic determined by the structure cycles on graphs possible transition return something condition to condition originally. A condition it is said periodic if factor fellowship the biggest from all over long track return worth bigger from one. On the other hand, a condition it is said aperiodic if factor fellowship the biggest the worth one. Existence bracelet (self-loop) is sufficient conditions for ensure aperiodic properties, although characteristic this can also happen without bracelet if graph transition allows track return with long mutual steps relatively prime.

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